Section 14.6
The Chain Rule

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1 The Chain Rule for 2-Variable Functions

The Chain Rule

Calculus of 1-variable: If $y = f(x)$ and $x = g(t)$, then the composition $y = (f \circ g)(t) = f(g(t))$ is a function of t. The Chain Rule says that

$$y'(t) = f'(g(t))\, g'(t) \qquad \text{or equivalently} \qquad \frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}.$$

Calculus of Multivariables: What about functions of more variables? For example, if

$$x = x(t), \qquad y = y(t), \qquad \text{and} \qquad z = f(x, y)$$

so that

$$z = f(x(t), y(t))$$

then, in principle, the derivative $z'(t)$ should depend on the derivatives

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}.$$

The Multivariable Chain Rule

Multivariable Chain Rule — First Case: Suppose that $z = f(x, y)$, $x = x(t)$, and $y = y(t)$ are differentiable functions. Then $z = g(t) = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

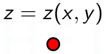
In Lagrange notation, the formula is

$$g'(t) = f_x(x, y)x'(t) + f_y(x, y)y'(t)$$

Idea: Changing t causes both x and y to change, which each cause changes in z. The Chain Rule formula records the total change in z due to a change in t, with contributions from both x and y.

The Chain Rule Tree

Let $z = z(x, y)$, $x = x(t)$, $y = y(t)$.
The pieces of the formula for dz/dt correspond to branches in a tree.



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[1] The first version of the tree was coded by Prof. Martin.

2 The chain Rule for Multivariable Functions

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The Multivariable Chain Rule

Example 1: Calculate dz/dt if

$$z(x,y) = xy, \qquad x = x(t) = \cos(t), \qquad y = y(t) = \sin(t).$$

Solution: The Chain Rule states that

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\,\frac{dx}{dt} + \frac{\partial z}{\partial y}\,\frac{dy}{dt} \\ &= (y)\,(-\sin(t)) + (x)\,(\cos(t)) \\ &= -\sin^2(t) + \cos^2(t)\end{aligned}$$

To confirm, note that z is a function of the single variable t:

$$z(t) = z\big(x(t), y(t)\big) = \cos(t)\sin(t)$$

By the Product Rule,

$$\frac{dz}{dt} = -\sin^2(t) + \cos^2(t)$$

Units in the Chain Rule

Let $z = z(x, y)$, $x = x(t)$, $y = y(t)$, so that the Chain Rule says

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

The units of this equation make sense. For example, if

$t =$ time (hours) $\qquad x =$ length (meters)

$y =$ temperature ($^\circ$C) $\qquad z =$ number of bananas

then the units of the Chain Rule equation are

$$\frac{\text{bananas}}{\text{hour}} = \frac{\text{bananas}}{\text{meter}}\,\frac{\text{meters}}{\text{hour}} + \frac{\text{bananas}}{^\circ\text{C}}\,\frac{^\circ\text{C}}{\text{hour}}$$

Why The Chain Rule Works (Optional)

Suppose that $x = x(t)$, and $y = y(t)$ are differentiable at $t = t_0$ and $f(x, y)$ is differentiable at $(a, b) = (x(t_0), y(t_0))$. By the definition of derivative,

$$\left.\frac{dz}{dt}\right|_{t=t_0} = \lim_{\Delta t \to 0} \frac{f\left(x(t_0 + \Delta t), y(t_0 + \Delta t)\right) - f(x(t_0), y(t_0))}{\Delta t}.$$

Let $x_1 = x(t_0 + \Delta t)$, $y_1 = (t_0 + \Delta t)$ and $\Delta x = x_1 - a$ and $\Delta y = y_1 - b$.

$$= \lim_{\Delta t \to 0} \frac{f(x_1, y_1) - L_{(a,b)}(x_1, y_1)}{\Delta t} + \lim_{\Delta t \to 0} \frac{L_{(a,b)}(x_1, y_1) - f(a, b)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(\frac{f(x_1, y_1) - L_{(a,b)}(x_1, y_1)}{\|(x_1, y_1) - (a, b)\|}\right) \left(\frac{\|(x_1, y_1) - (a, b)\|}{\Delta t}\right) + \lim_{\Delta t \to 0} \frac{L_{(a,b)}(x_1, y_1) - f(a, b)}{\Delta t}$$

Note that $\Delta x \to 0$ and $\Delta y \to 0$ as $\Delta t \to 0$ because x and y are differentiable, hence continuous, functions of t. So this limit becomes

$$= \underbrace{\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{f(x_1, y_1) - L_{(a,b)}(x_1, y_1)}{\sqrt{\Delta x^2 + \Delta y^2}}\right)}_{=0,\text{ since } f \text{ is diff.}} \underbrace{\lim_{\Delta t \to 0} \left(\frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta t}\right)}_{\text{exists, since } x,\, y \text{ are cont.}} + f_x(a, b) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + f_y(a, b) \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

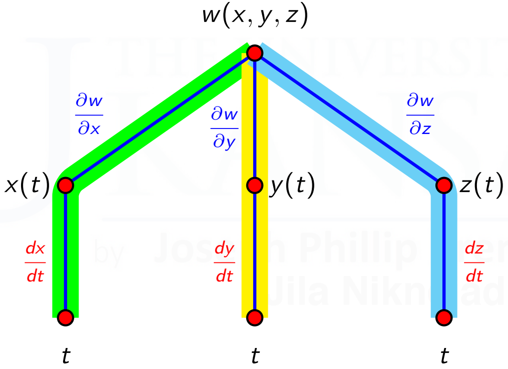
$$= f_x(a, b)\frac{dx}{dt} + f_y(a, b)\frac{dy}{dt}$$

▸ Sketch of the Proof

3 Chain Rule Dependency Tree Diagram

The Chain Rule Tree

Let $w = f(x, y, z)$, $x = x(t)$, $y = y(t)$, $z = z(t)$.
Again, the Chain Rule formula for dw/dt can be represented by a tree.



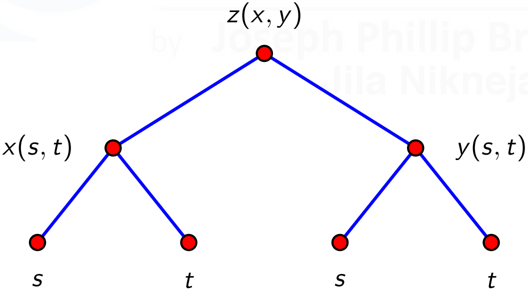
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

The Chain Rule: Another Case

For every composite function involving multiple variables, we can keep track of the Chain Rule using a tree.

Suppose $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, with all functions differentiable. Then $z(s, t) = f\left(g(s, t), h(s, t)\right)$ is differentiable, and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Example 2: Find $\partial z/\partial s$ and $\partial z/\partial t$ if

$$z = x^2 y + 2xy^4, \qquad x = st^2, \qquad y = s^2 t.$$

<u>Solution:</u>

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
&= (2xy + 2y^4)(t^2) + (x^2 + 8xy^3)(2st) \\
&= (2s^3t^3 + 2s^8t^4)(t^2) + (s^2t^4 + 8s^7t^5)(2st)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
&= (2xy + 2y^4)(2st) + (x^2 + 8xy^3)(s^2) \\
&= (2s^3t^3 + 2s^8t^4)(2st) + (s^2t^4 + 8s^7t^5)(s^2)
\end{aligned}$$

Tabular Chain Rule

Example 3: Consider $f(u,v) = g\big(x(u,v)\,,\; y(u,v)\big)$ where

$x(3,4) = 1$	$x_u(3,4) = -3$	$x_v(3,4) = -1$
$y(3,4) = 5$	$y_u(3,4) = 4$	$y_v(3,4) = 6$
$g(3,4) = 7$	$g_x(3,4) = 3$	$g_y(3,4) = 2$
$g(1,5) = -4$	$g_x(1,5) = 2$	$g_y(1,5) = -1$

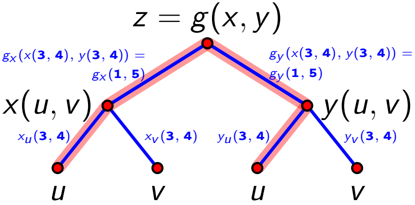
Compute $f_u(3,4)$.

Solution

$$f_u(3,4)$$

$$= g_x(1,5)x_u(3,4) + g_y(1,5)y_u(3,4)$$

$$= (2)(-3) + (-1)(4) = -10$$



4 The General Chain Rule

The General Chain Rule

The Multivariable Chain Rule—General Case: Suppose that $f(x_1, x_2, \ldots, x_n)$ is a function of n variables, and $x_1, x_2, \ldots, x_n$ are functions of m independent variables $t_1, t_2, \ldots, t_m$. Then

$$\frac{\partial f}{\partial t_j} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_j}$$

for each $j = 1, \ldots, m$.

The equations for all j can be expressed simultaneously in matrix form:

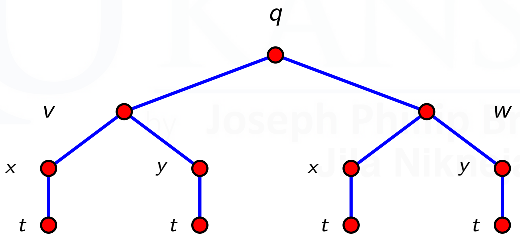
$$\begin{bmatrix} \frac{\partial f}{\partial t_1} & \cdots & \frac{\partial f}{\partial t_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_m} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_m} \end{bmatrix}$$

Triple (Or More) Compositions

The tree technique works even for compositions of three or more functions. For example, suppose that

$$x = x(t), \quad y = y(t), \quad v = v(x, y), \quad w = w(x, y), \quad q = q(v, w).$$

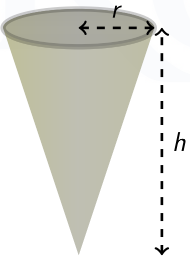
so that $q = q\Big(v\big(x(t), y(t)\big), w\big(x(t), y(t)\big)\Big)$.



$$\frac{dq}{dt} = \frac{\partial q}{\partial v}\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial q}{\partial v}\frac{\partial v}{\partial y}\frac{dy}{dt} + \frac{\partial q}{\partial w}\frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial q}{\partial w}\frac{\partial w}{\partial y}\frac{dy}{dt}$$

Related Rates and the Multivariable Chain Rule

Example 4: The volume of a cone is $V(r,h) = \dfrac{\pi r^2 h}{3}$, where r and h are the radius and the height of the cone. Both r and h are changing over time. What is $\dfrac{dV}{dt}$ at an instant when $r = 5$, $\dfrac{dr}{dt} = 3$, $h = 6$, and $\dfrac{dh}{dt} = -3$?



Solution (using the multivariable chain rule):

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{3}(2rh)\frac{dr}{dt} + \frac{\pi}{3}r^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi(2)(5)(3)(6)}{3} + \frac{\pi(5^2)(-3)}{3} = \boxed{35\pi}$$

5 Implicit Differentiation with the Chain Rule

Implicit Differentiation in Multiple Variables

If $z = z(x, y)$ is defined implicitly by an equation $F(x, y, z) = c$, then we can use the Chain Rule to find the partials $z_x = \partial z/\partial x$ and $z_y = \partial z/\partial y$.

Example 5: Suppose that $xz^2 + y^2z + xy = 1$. Find $\partial z/\partial x$.

Solution: Method 1: Differentiate both sides with respect to x:

$$\left(z^2 + 2xzz_x\right) + \left(2yy_xz + y^2z_x\right) + (xy_x + y) = 0$$

Note that $\boldsymbol{y_x = \partial y/\partial x = 0}$, because y is a separate independent variable whose value is not affected by changes in x. (Alternatively, y is fixed when we compute partials with respect to x.) So

$$z^2 + 2xzz_x + y^2z_x + y = 0 \qquad \Longrightarrow \qquad \boxed{z_x = \frac{-(z^2 + y)}{2xz + y^2}}$$

or Method 2:

$$z_x = \frac{-F_x}{F_z} = \frac{-(z^2 + y)}{2xz + y^2}$$

Implicit Differentiation in Multiple Variables

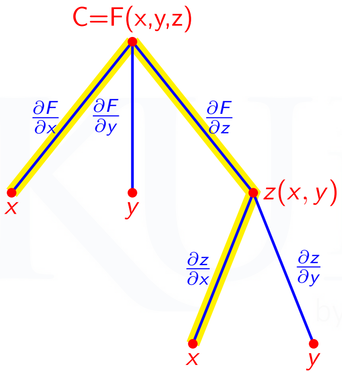
if $z = z(x, y)$ is defined implicitly by an equation $F(x, y, z) = c$, then the general formulas (which you can memorize if you so choose) are:

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \qquad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

You don't need to memorize these formulas, since they both come from the same process: differentiate both sides of the original equation and solve for the partial you are interested in.

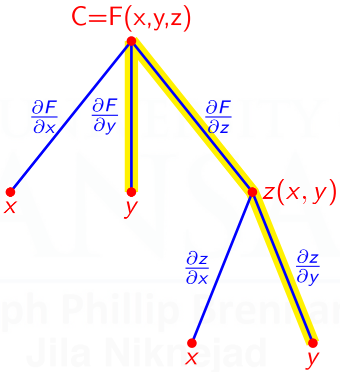
- Observation: $\partial z/\partial x$ and $\partial x/\partial z$ are reciprocals.
- The units in these formulas make sense (try it).

How to Memorize the Formula



$$0 = F_x(x, y, z) + F_z(x, y, z)\frac{\partial z}{\partial x}$$

Solve for $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$



$$0 = F_y(x, y, z) + F_z(x, y, z)\frac{\partial z}{\partial y}$$

Solve for $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$

Implicit Differentiation in Multiple Variables

Example 6: What is the slope of the surface given by $xy^2 + x^3z = z^3 + 1$ at the point $(1, 1, 0)$ in the direction given by the unit vector $\vec{u} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$?

Solution: The answer is the directional derivative $D_{\vec{u}}f(1, 1)$, where $z = f(x, y)$ is the function defined implicitly by the given equation.

Differentiate with respect to x:	Differentiate with respect to y:
$y^2 + (3x^2z + x^3z_x) = 3z^2z_x$	$2xy + x^3z_y = 3z^2z_y$
$z_x = \dfrac{y^2 + 3x^2z}{3z^2 - x^3}$	$z_y = \dfrac{2xy}{3z^2 - x^3}$
$z_x(1, 1, 0) = -1$	$z_y(1, 1, 0) = -2$

Answer: $\nabla f(1, 1) \cdot \vec{u} = \langle -1, -2 \rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle = 1.$